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LETTER TO THE EDITOR

Phase transition in a gauge model on a tree-like lattice

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Abstract. A new model which represents the gauge analogue of a spin system on a Cayley tree is proposed. The case of Z(2) symmetry is considered explicitly. The free energy exhibits singular behaviour at low temperatures as a function of a suitable symmetry breaking coupling. The Wegner–Wilson loop correlation function is shown to have an area and a perimeter type of decay in the high- and low-temperature phases respectively.

A possible Bethe-like approximation for the transition coupling of gauge systems is also proposed in connection with the model.

Lattice gauge models are important in different physical contexts. The study of phase transitions in non-Abelian theories is essential for the explanation of quark confinement and asymptotic freedom in hadron dynamics (Wilson 1974, Kogut 1979). On the other hand, Z(2) lattice gauge theory is relevant in condensed matter problems involving random interactions and frustration (Toulouse 1977).

In the present letter we propose and solve a model on a lattice, which can be considered as the gauge analogue of a Cayley tree (or Bethe lattice) in the case of spin systems (Domb 1960).

For many problems the solution on a tree is much simpler than on regular lattices. The behaviour of systems on trees, however, is not trivial, and is somehow related to the self-consistent Bethe–Peierls approximation (Eggarter 1974). Thus models on tree lattices also give 'classical' or 'mean-field' information on more realistic problems. This feature is particularly interesting in all cases for which the application of standard molecular field ideas is questionable. Gauge theories are certainly among these cases (Elitzur 1975, Pearson 1981).

The most relevant properties of Ising spin systems on Bethe lattices have been clarified and determined by Eggarter (1974) and by Müller-Hartmann and Zittartz (1974). More recently the work of these authors has been extended to spin systems with other global Abelian symmetries, like Potts, Z(N) or planar rotator models (Moraal 1981). A common property of these models is the existence of a high- and a low-temperature phase, separated by the Bethe–Peierls critical temperature (Bethe 1935, Domb 1960). In the low-temperature phase, the free energy exhibits singular behaviour in its external field dependence at zero field (Müller-Hartmann and Zittartz 1974).

Our lattice is made up of planar strips of square plaquettes, the strips having in common each longitudinal side (except those at the boundary) with other b ($b = 2, 3, \ldots$) strips. In figure 1 a schematic drawing of the lattice is given for the case

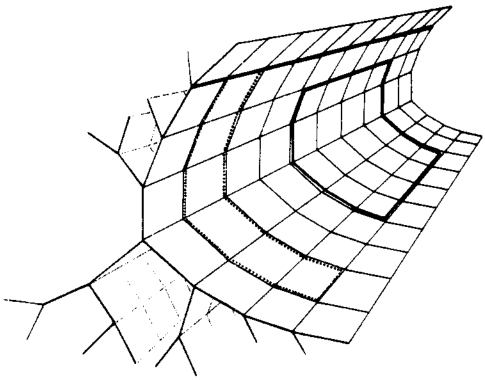

Figure 1. Our lattice for the case $b = 2$. The heavy longitudinal row supports the extrema of the C paths of equation (4). One of these paths is given in the figure (dotted lines). A 'rectangular' closed path is also drawn in the figure (heavy lines).

$b = 2$. The important topological feature of this lattice is that there is only one surface of plaquettes having a given closed path of bonds as the boundary. Of course, in the case $b = 1$ we obtain the two-dimensional square lattice.

We choose to work with Z(2) gauge theory for simplicity; the results we obtain below can be generalised without major difficulties to other discrete or continuous symmetries, as will be reported elsewhere.

Denoting by $\sigma_{ij}(=\pm 1)$ the variable associated with the nearest-neighbour bond (ij), the reduced Hamiltonian (action) of our system can be written as

$$-\beta\mathscr{H}(\sigma) = K \sum_{(ijkl)p} \sigma_{ij}\sigma_{jk}\sigma_{kl}\sigma_{li} \tag{1}$$

where $\beta = 1/k_{\mathrm{B}}T$, and the sum is extended to all sets of vertices of elementary oriented plaquettes. The Hamiltonian (1) is invariant under gauge transformations of the form

$$\sigma_{ij} \to \sigma'_{ij} = \gamma_i \sigma_{ij} \gamma_j^{-1} \tag{2}$$

where γ_l, at site l, takes the values ± 1.

Let us consider a finite lattice of the type illustrated in figure 1, with open boundary conditions. If we choose an arbitrary longitudinal row of bonds (heavy line in figure 1) in the lattice, it is immediate to verify that the partition function Z satisfies

$$Z = \mathrm{Tr}_{\{\sigma\}} \, \mathrm{e}^{-\beta\mathscr{H}(\sigma)} = \mathrm{Tr}_{\{\gamma\}} \, \mathrm{Tr}_{\{\sigma\}} \, \mathrm{e}^{-\beta\mathscr{H}(\sigma)} \prod_{(ij)\mathrm{transv.}} 2^{-1}(1+\gamma_i\sigma_{ij}\gamma_j^{-1}) \tag{3}$$

where the product runs over all bonds, which are not parallel to the given row (transverse); the γ's on the sites of the row itself are fixed equal to $+1$, and there is no summation on them†. Equation (3) and the gauge invariance of the Hamiltonian imply that up to an inessential numerical factor ($=\mathrm{Tr}_{\{\gamma\}}1$), the partition function factorises into the product of M partition functions for independent spin systems on Cayley trees, M being the number of bonds on a longitudinal row. Indeed, for each configuration $\{\gamma\}$ in (3), one can induce a corresponding gauge transformation (2), leading to a system where all the transverse σ's are fixed to be $+1$. In this way only

† This last restriction will be necessary in the following discussion, when we also consider symmetry breaking interactions.

the longitudinal σ's fluctuate and interact with bilinear, spin–spin-like interactions of strength K, between bonds on the same plaquette. Up to a constant the free energy per bond of our system in the thermodynamic limit ($M \to \infty$ and all transverse branches extend to infinity with respect to a given longitudinal line) is equal to $\frac{1}{2}$ of the free energy per site of a spin model on a Cayley tree; the latter free energy is known to be analytic in K, for all K, at zero external field h. For such spin systems, only the derivatives of the free energy with respect to an external magnetic field become singular below the Bethe–Peierls critical temperature ($K_{\mathrm{BP}} = \tanh^{-1}(1/b)$ for the Ising case) (Eggarter 1974, Müller-Hartmann and Zittartz 1974); the field singularities are accompanied by spontaneous symmetry breaking in the 'interior' of the system†. This symmetry breaking means that, for $K > K_{\mathrm{BP}}$, $\lim_{h \to 0\pm} \lim_{N \to \infty} \langle \sigma_0 \rangle_N \neq 0$, where σ_0 is in the interior and N is its minimal distance from the boundary.

In order to see any singular behaviour in our gauge model, we must first add to (1) a symmetry breaking interaction, which possibly induces an effect analogous to that of the magnetic field in the case of the spin tree. For this purpose we choose to introduce an interaction which is not gauge invariant and has the form

$$h \sum_{C} \prod_{(i,j) \in C} \sigma_{ij} \tag{4}$$

where C indicates any possible connected open path starting and ending at the two extrema of a bond belonging to an arbitrarily chosen longitudinal row. Each path C contains only one longitudinal bond, and thus has a strip-like rectangular shape. C coincides with the longitudinal bond when this belongs to the chosen row in the lattice. A path of this type is drawn in figure 1. By choosing the particular longitudinal row in equation (3) coincident with the row where all C paths start and end, it is easy to verify that the partition function of our gauge model, with the interaction (4) added to (1), factorises into the product of M partition functions for Cayley spin trees in an external field h. So, in this case, the free energy per bond is (up to a constant and a factor $\frac{1}{2}$) that of a Cayley spin tree with NN interaction K in an external field h.

The singularities in h of the free energy of this spin model have been already computed and turn out to correspond to a transition of an order varying continuously between first and infinite order for a range of K between ∞ and K_{BP} (Müller-Hartmann and Zittartz 1974). In the present work we perform a calculation of the spin–spin correlation function; as we will see, this calculation allows us to discuss the behaviour of the Wegner–Wilson loop correlation function for the gauge model. This is a gauge-invariant correlation function, $W_\Gamma = \langle \Pi_{(ij) \in \Gamma} \sigma_{ij} \rangle$ with Γ being a closed loop, whose behaviour for large Γ should characterise the possible different phases of the system (Wegner 1971, Wilson 1974).

Using equations (3) and (2) it is easy to verify, for example, that for a 'rectangular' Γ of lengths m and n, in the longitudinal and transverse directions respectively, (see figure 1)

$$W_\Gamma = \langle \sigma_0 \sigma_n \rangle^m \tag{5}$$

where $\langle \sigma_0 \sigma_n \rangle$ is the correlation function of two interior spins, at distance n on a spin tree, in the thermodynamic limit. As we show below, the calculation of $\langle \sigma_0 \sigma_n \rangle$ can be performed by making reference to the iterative method already used for computing the free energy and the core magnetisation of such spin systems (Eggarter 1974,

† A point is in the interior if, in the thermodynamic limit, all paths connecting it to boundary spins become infinite.

Müller-Hartmann and Zittartz 1974). Other techniques for calculation of $\langle\sigma_0\sigma_n\rangle$ are not able to give the correlation in the presence of an external field. For example, the results of Wang and Wu (1976) for the Potts model are appropriate only for h strictly equal to zero without any symmetry breaking effects. In the iterative method one considers the effective field x_N felt by the top spin of an N-generation branch of the system†, and finds a recursion giving x_{N+1} in terms of x_N. For our Ising spin trees one simply gets (Müller-Hartmann and Zittartz 1974)

$$x_{N+1}=h+b\ \ln\left[(\exp(2x_N)+\exp(-2K))/(\exp(2x_N-2K)+1)\right]. \tag{6}$$

At $h=0$ and for $K<K_{\rm BP}$, equation (6) has only a stable fixed point $x^*=0$, which becomes marginal for $K=K_{\rm BP}$. For $h=0$ and $K>K_{\rm BP}$, $x^*=0$ becomes unstable, and two symmetric attractive fixed points $x^*=\pm h_\infty(K)\neq 0$ appear. The appearance of these last fixed points is the mechanism explaining the previously mentioned spontaneous symmetry breaking in the interior of the system. If $K>K_{\rm BP}$, performing the thermodynamic limit ($N\to\infty$) with $h>0$, and then letting $h\to 0$, will leave an effective non-zero field acting on each spin in the interior. This field is equal to $\tanh^{-1}(\tanh K\ \tanh h_\infty)+h_\infty$ because a spin in the interior can be seen as the top of an infinite-generation branch, interacting via K with the top spin of another infinite-generation branch. The internal spontaneous magnetisation is thus non-zero for $K>K_{\rm BP}$, and is given by

$$\langle\sigma_0\rangle\equiv\lim_{h\to0+}\lim_{N\to\infty}\langle\sigma_0\rangle_N=\frac{\tanh h_\infty(1+\tanh K)}{1+\tanh K\ \tanh^2 h_\infty}=m_0>0. \tag{7}$$

The calculation of $\langle\sigma_0\sigma_n\rangle$ on the spin Cayley tree can be performed by considering the path connecting σ_0 with σ_n. As one can argue from the drawing in figure 2, in the thermodynamic limit the intermediate spins along the path, $\sigma_1,\sigma_2,\ldots$ and σ_{n-1}, for $K>K_{\rm BP}$ feel a non-zero effective field, even after letting $h\to 0+$. This field comes from the interaction with the top spin of an infinite-generation branch, and is equal to $\tanh^{-1}(\tanh K\ \tanh h_\infty)$. At the same time, the extremal spins σ_0 and σ_n are the top spins of infinite-generation branches and feel a field h_∞ for $h\to 0+$. Thus the problem of obtaining $\langle\sigma_0\sigma_n\rangle$ in the thermodynamic limit is reduced to that of computing the correlation between the extremal spins of a one-dimensional Ising chain of length n

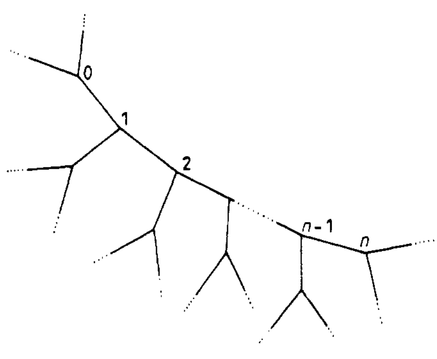

Figure 2. Schematic representation of the path connecting sites 0 and n in the Bethe lattice (with $b=2$).

† This is defined as an initial (top) site, connected with b $(N-1)$-generation branches; the one-generation branch is a single site.

with suitable external fields. This can be done easily by the transfer matrix method (Schultz *et al* 1964). So, for $K < K_{BP}$ and $h = 0$, $\langle \sigma_0 \sigma_n \rangle = (\tanh K)^n$; on the other hand, for $K > K_{BP}$, if h approaches zero in the infinite system, we get a different correlation, which does not tend to zero as $n \to \infty$; in this limit the correlation factorises into the square of (7), as one can verify after some tedious algebra. In view of equation (5), these results immediately determine W_Γ and its behaviour for large Γ. In the high-temperature region ($K < K_{BP}$) the absence of spontaneous symmetry breaking in the interior of the spin trees implies

$$W_\Gamma \underset{m,n \to \infty}{\simeq} \exp(-|\ln \tanh K|mn) \tag{8}$$

which is the area decay law expected in gauge systems at high temperature (Wegner 1971, Wilson 1974).

For low temperatures ($K > K_{BP}$) if we let our symmetry breaking interaction (4) go to zero, we find the behaviour

$$W_\Gamma \underset{m,n \to \infty}{\simeq} \exp(-|\ln m_0|2m) \tag{9}$$

with m_0 given by equation (7). This is not exactly the perimeter type of decay expected in the low-temperature phase of a gauge system with phase transition (Wegner 1971, Wilson 1974); it is, indeed, even slower than a perimeter decay, for which the exponent should be proportional to $2(m+n)$. The anisotropic character of equation (9) is of course connected with the peculiar topology of our system, in which longitudinal and transverse directions play radically different roles, as far as the mechanism leading to (9) is concerned. Our method of calculation of W_Γ is very general and applies to every conceivable closed Γ in the interior of the lattice.

At this point a remark concerning the possibility of local symmetry breaking in our gauge system is in order. When we add the interaction (4) to equation (1), we can show that, according to Elitzur's theorem (Elitzur 1975), one must have $\lim_{h\to 0} \langle \sigma_{ij} \rangle = 0$ for all the bonds in our system which do not belong to the particular longitudinal row supporting the extrema of the open paths C of (4). For these last bonds the structure of equation (4), however, does not exclude the possibility of local symmetry breaking, that is one can have $\lim_{h\to 0\pm} \langle \sigma_{ij} \rangle \neq 0$. Actual calculation of $\langle \sigma_{ij} \rangle$ for these bonds leads to the result that local symmetry breaking takes place, for $K > K_{BP}$, only if the longitudinal row is chosen in the interior of the system, that is, if it stays infinitely far from the boundaries in the thermodynamic limit. On the contrary, if we choose this row right on the boundary for example, no local spontaneous symmetry breaking occurs. This result is connected with the fact that, in a spin tree, spontaneous magnetisation develops only in the core (Eggarter 1974).

In conclusion we have proposed a gauge model which can be solved exactly and, like spin models on Cayley trees, has a phase transition; this transition is detectable in the dependence of the free energy on the coupling h of the symmetry breaking interaction (4). In the high-temperature region the asymptotic behaviour (8) of W_Γ indicates a 'confining' phase (Wilson 1974). The asymptotic behaviour (9), on the other hand, shows the 'deconfining' character of the low-temperature phase.

As for spin trees, the interest of this model is primarily due to its solvability, combined with the presence of non-trivial behaviour; these qualities should be most appreciated in the context of gauge theories, where exact solutions of more realistic models do not exist.

We finally come to the above mentioned connection between systems on trees and classical approximations.

The critical temperature of a spin system on a Bethe lattice with coordination number $b+1$ is the same (Eggarter 1974) as the one given by the Bethe–Peierls approximation (Bethe 1935, Domb 1960) for a hypercubic lattice in $d=\frac{1}{2}(b+1)$ dimensions. For gauge models Bethe-like approximations have not yet been proposed. The results of our model, however, allow us to guess, by analogy, what could be the result for the critical coupling of such a type of classical approximation in the gauge case. Each bond in a hypercubic lattice belongs to $2(d-1)$ plaquettes; in our model, on the other hand, a longitudinal bond belongs to $(b+1)$ plaquettes, thus the approximate Bethe–Peierls like critical coupling should be

$$K_{\mathrm{BP}} = \tanh^{-1}(2d-3)^{-1} \tag{10}$$

which correctly goes to infinity for $d \to 2$, since we know that the two-dimensional model has no phase transition at finite temperature (Kogut 1979).

When obtaining equation (10), we did not consider the coordination of transverse bonds in our system. At first sight this could appear somewhat arbitrary. There is, however, a convincing argument in favour of our choice. Indeed, for Z(2) (and in general Abelian) gauge theory on a d-dimensional hypercubic lattice, the following inequality has been proved (Brydges *et al* 1979):

$$W_\Gamma \leqslant \langle \sigma_0 \sigma_n \rangle^m \tag{11}$$

where Γ is again a rectangular $m \times n$ loop, and $\langle \sigma_0 \sigma_n \rangle$ is a correlation in the $(d-1)$-dimensional spin system. Thus for our model an analogous type of inequality (see equation (5)) is satisfied as an equality. In our case the reduction of dimensionality is due to the disappearance of the longitudinal lattice direction when passing from the gauge to the spin tree. Coming back to equation (10), which is based on the position $(b+1)=2(d-1)$, we notice that this is the only choice leading to a difference of 1 between the d for the gauge model and the d for the spin model. The latter one, indeed, is obtained from $(b+1)=2d$, our gauge and spin trees having the same b.

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